A brief introduction to non-dimensionalization

Consider the following equation:

\[ m_1 \left( 2r \dot{\theta} + r^2 \ddot{\theta} \right) + \frac{h^2}{m_1 r^2} = m_2 g (c^2 - r) \sin(\theta) - k(r - r_0)^2, \quad (1) \]

where

\[ m_1, m_2, r_0, c, g \text{ and } h(= m_1 (c^2 - r)^2 \sin^2(\phi) \dot{\theta}) \quad (2) \]

are constants. Now, suppose the Secret Service comes knocking on your door in the middle of the night, demanding that you non-dimensionalize Eqn. 1 due to your enviable intellectual capabilities (and the fact that they have, somehow, determined that you are enrolled in ME175). This is my attempt at providing you with a rough guide for non-dimensionalizing a given equation because you can never tell when the Secret Service might come knocking...

**Step 1**
Write down the given constants and their units. Thus, for the problem given, we have:

<table>
<thead>
<tr>
<th>Constant</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1, m_2)</td>
<td>([M])</td>
</tr>
<tr>
<td>(r_0)</td>
<td>([L])</td>
</tr>
<tr>
<td>(c)</td>
<td>([L]^\frac{1}{2})</td>
</tr>
<tr>
<td>(g)</td>
<td>([L][T]^{-2})</td>
</tr>
<tr>
<td>(k)</td>
<td>([M][T]^{-2})</td>
</tr>
<tr>
<td>(h)</td>
<td>([M][L]^2[T]^{-1})</td>
</tr>
</tbody>
</table>

**Step 2**
Write down the variables involved and their units:

<table>
<thead>
<tr>
<th>Variables</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>([T])</td>
</tr>
<tr>
<td>(r)</td>
<td>([L])</td>
</tr>
<tr>
<td>(\theta, \phi)</td>
<td>radian</td>
</tr>
</tbody>
</table>

Note that we assume that the radian unit is already dimensionless.
**Step 3**
Determine a new set of non-dimensional variables based on the variables present in
the problem. Thus, to non-dimensionalize \( t \), we would need to multiply it by some
combination of constants that has units of \( \frac{1}{[T]} \). Looking at table 1, we find that the
following combinations (among many others) have units of \( \frac{1}{[T]} \):
\[
\sqrt{g/c^2}, \sqrt{g/r_0}, \sqrt{k/m_1}, \sqrt{k/m_2}, \sqrt{gkr_0/h}, h/(m_1c^4), \text{ etc}
\]
Any one of this would be absolutely correct choices. The main reason we select one
over the other is because we hope to be able to factor out similar terms in the end. For
this problem, I decided to use
\[
\tau = \sqrt{\frac{k}{m_1}}t \quad \text{and} \quad \bar{r} = \frac{r}{r_0} \quad (\text{or} \ r = \bar{r}r_0)
\]
as the non-dimensional variables. I selected \( m_1 \) instead of \( m_2 \) in the denominator since
\( m_1 \) appears twice as often as \( m_2 \) in the given equation. Choosing \( m_2 \) won’t result in a
wrong equation. Instead, it will probably result in a relatively more complicated equa-
tion compared to that obtained had we used \( m_1 \).  
(There is no wrong equation unless we have been given a wrong equation in the first place! All you are doing is divid-
ing/multiplying it by an intelligent choice of constants to nondimensionalize it...and to
appease the Secret Service, of course). Additionally, I didn’t need to non-dimensionalize
the angles \( \theta \) and \( \phi \) because, as mentioned previously, radians are assumed to be a non
dimensional unit.

**Step 4**
Calculate all the required derivatives in terms of the non-dimensionalized variables
and the constants of the equation. Recall that our equation was
\[
m_1 \left( 2\bar{r} \dot{\theta} + r^2 \ddot{\theta} \right) + \frac{h^2}{m_1 r^2} = m_2 g (c^2 - r) \sin(\theta) - k(r - r_0)
\]
. Thus, we see that
\[
\frac{dr}{dt} = \frac{d\bar{r}}{d\tau} \left( \frac{d\bar{r}}{dt} \right) = \frac{dr}{d\tau} \left( \frac{d\bar{r}}{d\tau} \frac{d\tau}{dt} \right) = r_0 \left( \frac{d\bar{r}}{d\tau} \sqrt{\frac{k}{m_1}} \right)
\]
\[
\frac{d\theta}{dt} = \frac{d\theta}{d\tau} \frac{d\tau}{dt} = \frac{d\theta}{d\tau} \sqrt{\frac{k}{m_1}}
\]
\[
\frac{d^2\theta}{dt^2} = \frac{d}{dt} \left( \frac{d\theta}{dt} \right) = \frac{d}{d\tau} \left( \frac{d\theta}{d\tau} \sqrt{\frac{k}{m_1}} \right) = \frac{d}{d\tau} \left( \frac{d\theta}{d\tau} \sqrt{\frac{k}{m_1}} \frac{d\tau}{d\tau} \right) = \frac{d^2\theta}{d\tau^2} \frac{k}{m_1}
\]
Notice that every time derivative introduces a factor of \( \sqrt{\frac{k}{m_1}} \) since \( \frac{d}{dt} = \frac{d}{d\tau} \frac{d\tau}{dt} = \sqrt{\frac{k}{m_1}} \frac{d}{d\tau} \).
Thus,
\[
\frac{dr}{dt} = \sqrt{\frac{k}{m_1}} \frac{dr}{d\tau}; \quad \frac{d^2r}{dt^2} = \frac{k}{m_1} \frac{d^2r}{d\tau^2}; \quad \frac{d^3r}{dt^3} = \left( \frac{k}{m_1} \right)^{\frac{3}{2}} \frac{d^3r}{d\tau^3} \text{ etc.}
\]

**Step 5**
This is the moment of truth. Replace all the dimensional variables in
\[
m_1 \left( 2\bar{r} \dot{\theta} + r^2 \ddot{\theta} \right) + \frac{h^2}{m_1 r^2} = m_2 g (c^2 - r) \sin(\theta) - k(r - r_0)^2
\]
with their non-dimensional equivalents. Thus, replace \( r \) with \( \tilde{r}r_0 \), \( \dot{r} \) with \( r_0\sqrt{\frac{k}{m_1}} \frac{d\tau}{dt} \) and so forth to get

\[
2m_1\tilde{r}\dot{\theta} = 2m_1r_0\tilde{r} \left( r_0\sqrt{\frac{k}{m_1}} \frac{d\tilde{r}}{d\tau} \right) \left( \sqrt{\frac{k}{m_1}} \frac{d\theta}{d\tau} \right) = 2m_1r_0^2\tilde{r} \frac{k}{m_1} \frac{d\tilde{r}}{d\tau} \frac{d\theta}{d\tau} = 2r_0^2\tilde{r}k \frac{d\tilde{r}}{d\tau} \frac{d\theta}{d\tau},
\]

\[
m_1r^2\ddot{\theta} = m_1r_0^2\tilde{r}^2 \left( \frac{k}{m_1} \frac{d^2\theta}{d\tau^2} \right) = \frac{r_0^2\tilde{r}^2k}{m_1} \frac{d^2\theta}{d\tau^2},
\]

and

\[
m_2g(c^2 - r) \sin(\theta) - k(r - r_0) = m_2g(c^2 - r_0\tilde{r}) \sin(\theta) - k(\tilde{r}r_0 - r_0)^2
\]

\[
= m_2g \left( \left( \frac{c^2}{r_0} \right) r_0 - r_0\tilde{r} \right) \sin(\theta) - k(\tilde{r} - 1)^2r_0^2
\]

\[
= \left( m_2g \left( \frac{c^2}{r_0} - \tilde{r} \right) \sin(\theta) - kr_0(\tilde{r} - 1)^2 \right) r_0. \tag{5}
\]

Putting everything together, we get:

\[
2r_0^2\tilde{r}k \frac{d\tilde{r}}{d\tau} \frac{d\theta}{d\tau} + r_0^2\tilde{r}^2k \frac{d^2\theta}{d\tau^2} + \frac{h^2}{m_1r_0^2}\tilde{r}^2 = r_0 \left( m_2g \left( \frac{c^2}{r_0} - \tilde{r} \right) \sin(\theta) - kr_0(\tilde{r} - 1)^2 \right) \tag{6}
\]

Dividing the equation above by \( kr_0^2 \) (since they appear as common factors in the first and second terms on the left hand side and on the last term on the right hand side) and setting

\[
\alpha = \left( \frac{c^2}{r_0} \right), \quad \beta = \frac{m_2g}{kr_0}, \quad \text{and} \quad \tilde{h} = \frac{h^2}{m_1kr_0^2},
\]

we obtain

\[
2\tilde{r} \frac{d\tilde{r}}{d\tau} \frac{d\theta}{d\tau} + \tilde{r}^2 \frac{d^2\theta}{d\tau^2} + \tilde{h} \frac{d^2\theta}{d\tau^2} = \beta(\alpha - \tilde{r}) \sin(\theta) - (\tilde{r} - 1)^2, \tag{8}
\]

which has one less constant (4: \( r_0, \alpha, \beta, \text{ and } \tilde{h} \)) compared to the 7 (\( m_1, m_2, r_0, c^2, g, k \) and \( h \)) in our original equation,

\[
m_1 \left( 2\tilde{r}\dot{\theta} + \tilde{r}^2\ddot{\theta} \right) + \frac{\tilde{h}^2}{m_1r_0^2} = m_2g(c^2 - r) \sin(\theta) - k(r - r_0)^2
\]

If our initial equation had been simpler:

\[
m \left( 2\tilde{r}\dot{\theta} + \tilde{r}^2\ddot{\theta} \right) + \frac{\tilde{h}^2}{m_0^2} = -k(r - r_0)^2, \tag{9}
\]

then using the same non-dimensionalized variables as before results in the following equivalent equation:

\[
2\tilde{r} \frac{d\tilde{r}}{d\tau} \frac{d\theta}{d\tau} + \tilde{r}^2 \frac{d^2\theta}{d\tau^2} + \tilde{h} \frac{d^2\theta}{d\tau^2} = -(\tilde{r} - 1)^2. \tag{10}
\]

We no longer need to worry about \( k \) or \( m \). All we are left with is \( r_0 \) and \( \tilde{h} \) which incorporates the dependence of the equation on both \( m \) and \( k \). Further, note that while \( h \) was defined as

\[
h = m_1(c^2 - r)^2 \sin^2(\phi) \dot{\theta},
\]
we leave it is a constant throughout the non-dimensionalization because, as far as we are concerned, in the given equation, \( h \) is, in fact, just some arbitrary constant. The fact that \( h \) consists of the dimensional variables \( r, \theta \) and \( \phi \) does not matter because they multiply together to result in a constant. If you are still a little wary of this, look at the dimensions of \( \bar{h} \) in Eqn 7: it is non-dimensional:

\[
\frac{[h^2]}{[m_1kr_0^2]} = \frac{[M]^2[L]^4[T]^{-2}}{([M])([M][T]^{-2})([L]^4)} = 1.
\]

(11)

In Conclusion

While non-dimensionalizing might seem like a lot of work for very little gain, it is actually a very useful tool. It allows us to significantly simplify the analysis of a broad array of problems by reducing what may seem like completely disparate systems to systems that are governed by essentially the same rules of motion. For example, the simple pendulum and the spring mass system both have similar phase portraits in the range where \( \sin(\theta) \approx \theta \).

Furthermore, there are times when numerically integrating a given equation might take longer than a single semester due to the sizes of some of the constants present. For example, in planetary motion, the masses are on the order of \( 10^{24} \text{kg} - 10^{30} \text{kg} \) and the units of time are on the order of \( 3 \times 10^7 \) seconds. Attempting to integrate one of those central force motion ode’s might take much longer than one solar orbit! Non-dimensionalizing the equations eliminates this problem because it eliminates some of the dependence on the constants, focusing instead on the ratio of constants. Thus, if two masses were both roughly \( 10^{24} \) kilograms (plus or minus a few kilograms, of course), then their ratios would be unity. Likewise, if we can find a constant with units of \( \frac{1}{[T]} \) that is on the order of \( 10^{-7} \left( \sqrt{\frac{GM_{\text{earth}}}{R_{\text{sun-earth-dist}}}^3} \approx 10^{-7} \right. \) for example), then the numerical equation can be solved in much less time, and is also less prone to instability.

Of course, there’s also the fact that the Secret Service may someday, request that you do it...